The IV Formulation and Linear Approximations of the AC Optimal Power Flow Problem

Optimal Power Flow Paper 2

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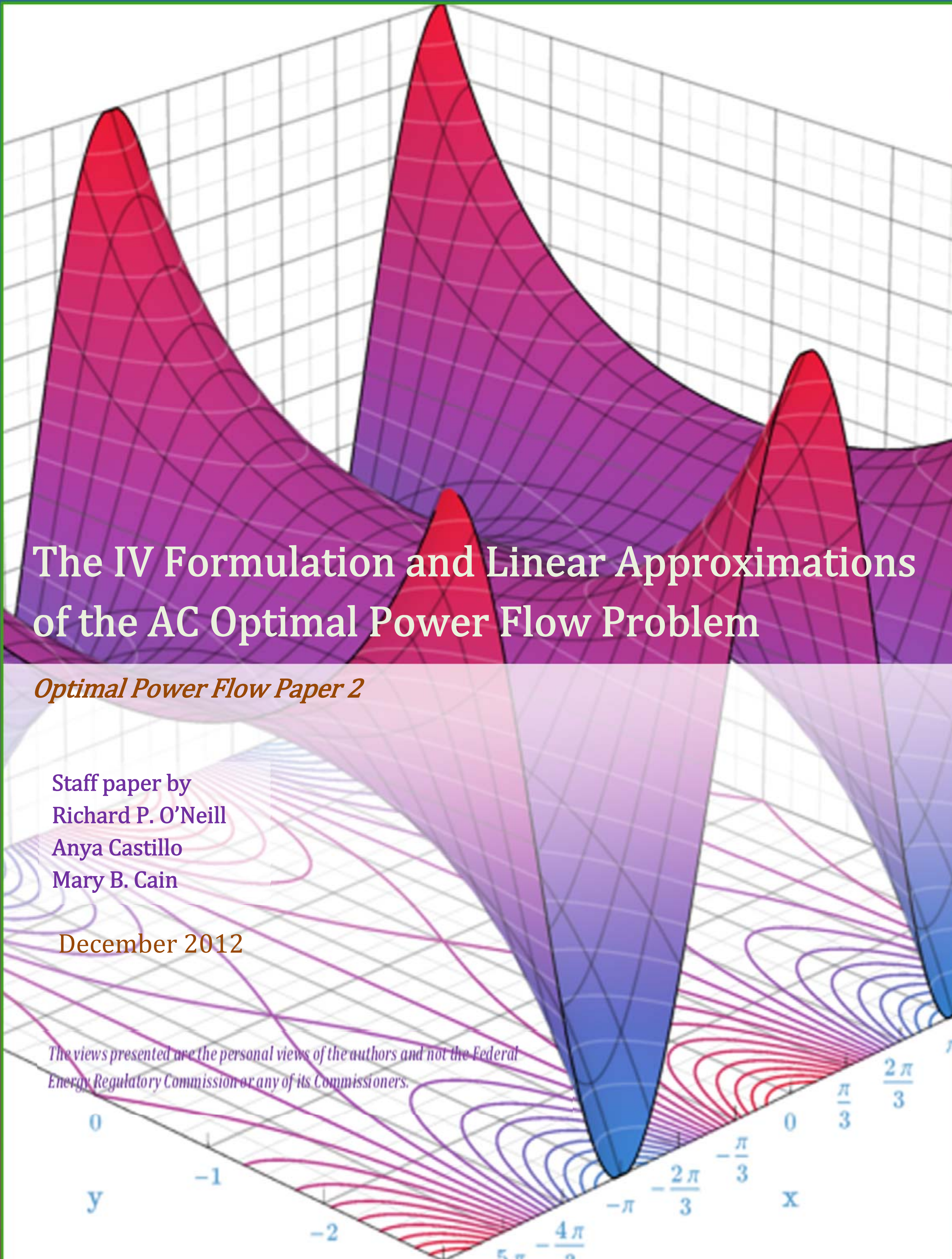

The IV Formulation and Linear Approximations of the AC Optimal Power Flow Problem

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Abstract and Executive Summary

Understanding AC Optimal Power Flow (ACOPF) is important because a one percent improvement in a power system dispatch can save roughly tens of billion dollars annually. In this paper, we formulate the ACOPF in several ways, compare each formulation's properties, and argue that the current-voltage (IV) formulation and its linear approximations may be easier to solve than the traditional quadratic power flow formulations. Unlike the DC model that holds voltage constant, ignores reactive power and assumes small voltage angle differences, the IV formulation solves a linear system of equations without decomposition, unnecessary constraints or omissions. The nonconvex constraints need careful consideration. For problems that are solved repetitively with minor variations, there is considerable potential for individual parameter tuning and preprocessed constraints. For example, constraints on angles and past iteratively added constraints may be able to help. Initial results indicate that the IV formulation and its linear approximations have promise to meet practical computational requirements. In addition, steady-state quadrature constraints on generators are linear in the IV formulation and can be included in the formulation. Additional formulation testing and computational testing are needed to determine commercial feasibility. If a linear IV approximation to the ACOPF proves promising, it can be embedded in the unit commitment models, optimal topology models and other formulations that use binary variables. This allows the use of mixed integer linear programs (MIP) algorithms that are exceptionally fast and robust to better model the power markets.

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1. Introduction

The Alternating Current Optimal Power Flow (ACOPF) problem is important—a one percent improvement in dispatch derived from better solutions to this problem can save roughly 1 to 5 billion dollars per year in the US, or 4 to 20 billion dollars per year in the world (see Cain 2012). In this paper, we formulate the ACOPF in several ways, compare each formulation's properties, and argue that the current-voltage (IV) formulation and its linear approximations may be easier to solve than the traditional quadratic power flow formulation.

We focus on the linear approximation because the linear program solvers are robust and fast and the power systems community is already familiar with linear programs for economic dispatch. The linear program approximation constructs linear relaxations of convex constraints and linear first order approximations to the non-convex constraints. If a linear IV approximation to the ACOPF proves promising, it can be embedded in the unit commitment models, optimal topology models and other formulations that use binary variables. This allows the use of mixed integer program (MIP) algorithms that are exceptionally fast and robust to better model the power markets. Since both MIP algorithms and computer hardware continue to improve, today's proof-of-concept software may be tomorrow's commercial standard. Power system optimization has evolved with improvements in computer hardware and optimization software. In companion papers, we present more detail on the formulations and on several test problems. We also examine some of the arguments presented here with success.

The optimal power flow (OPF) problem finds the optimal solution to an objective function subject to the power flow constraints. There are a variety of OPF formulations with different constraints, different objective functions, and different solution methods that have been labeled optimal power flow. The simplest optimal power flow model is known as the "Direct Current (DC) OPF". It uses a linearized approximation of the AC power flow equations and linear constraints. Formulations that use the AC power flow equations are known as "AC OPF." The ACOPF formulations are continuous nonconvex optimization problems without binary variables. For more detail, see Cain et al (2012). Most nonlinear solvers only find local optimal solutions for nonconvex problems.

2. Notation

When n and m are subscripts, they index buses; k indexes each three-phase transmission element between buses n and m. When j is not a superscript, $\boldsymbol{j} = (-1)^{1/2}$; the complex current is i. When j is a superscript, it is the 'imaginary' part of a complex number, while an r superscript is the 'real' part. Unless specified otherwise, scalars and complex numbers are lower case. Generally, in $(x^{y}{}_{z})^{w}$, x is a variable or parameter, y further defines the variable or parameter, z is a member of an index set, w is an exponent.

Unless specified otherwise, matrices are upper case. All vectors are column vectors and T superscript represents a transpose operation. The element-by-element or Hadamard product is •. If diag(A) is a square matrix with A on the diagonal and zeroes elsewhere, diag(A)diag(B) = diag(A•B). The complex conjugate operator is * (superscript) and * (no superscript) is an optimal solution.

We assume balanced three-phase steady-state conditions. All variables are associated with a single-line model of a balanced three-phase system. A common practice in power system modeling is the per-unit (p.u.) representation, where base quantities for voltage, current, power, and impedance (or admittance) may fully define a power network. Such normalization is a convenience.

The topology of the network consists of locations known as buses or nodes and transmission elements connecting paired buses. The network is an undirected graph with weighted edges or lines.

Indices and Sets

n, m are bus (node) indices; $n, m \in \{1, \ldots, N\}$ where n is the number of buses.

k is a three-phase transmission element index. Each transmission element k has a pair of terminal buses n and m. $k \in \{1, \ldots, K\}$ where k is the number of transmission elements

K' is the number of a connected bus pairs ($K' \leq K$).

Unless otherwise stated, summations ($\sum$) are over the full set of indices.

Variables

p_n is the real power injection (positive) or withdrawal (negative) at bus n

q_n is the reactive power injection or withdrawal at bus n

$s_n = p_n + \boldsymbol{j}q_n$ is the net complex power injection at bus n

p_{nmk} is the real power flow at bus n to bus m on transmission element k

q_{nmk} is the reactive power flow at bus n to bus m on transmission element k

θ_n is the voltage phase angle at bus n

$\theta_{nm} = \theta_n - \theta_m$ is the voltage phase angle differences from bus n to bus m

i_n is the current (complex phasor) injection (positive) or withdrawal (negative) at bus n where $i_n = i^r{}_n + \boldsymbol{j}i^j{}_n$

i_{nmk} is the current (complex phasor) flow into transmission element k at bus n (to bus m); $i_{nmk} = i^r{}_{nmk} + \boldsymbol{j}i^j{}_{nmk}$

s_{nmk} is the apparent complex power flow into bus n on transmission element k (to bus m); $s_{nmk} = s^r{}_{nmk} + \boldsymbol{j}s^j{}_{nmk}$

v_n is the complex voltage at bus n. $v_n = v^r{}_n + \boldsymbol{j}v^j{}_n$

y_{nmk} is the complex admittance on transmission element k connecting bus n and bus m (If buses n and m are not connected directly, y_{nmk}= 0.);

y_n is the self-admittance (to ground) at bus n.

y_{nm} is the complex admittance connecting bus n and bus m for all transmission elements k between buses n and m.

$V = (v_1, \ldots, v_N)^T$ is the complex vector of bus voltages; $V = V^r + jV^j$

$I = (i_1, \ldots, i_N)^T$ is the complex vector of bus current injections; $I = I^r + jI^j$

$P = (p_1, \ldots, p_N)^T$ is the vector of real power injections

$Q = (q_1, \ldots, q_N)^T$ is the vector of reactive power injections

G is the N-by-N conductance matrix

B is the N-by-N suseptance matrix

Note that elements of G and B will be constant for passive transmission elements such as transmission lines, but can be variable when active transmission elements such as phase shifting transformers, switched capacitors/reactors, or FACTs devices are included.

$Y = G + jB$ is the complex admittance N-by-N matrix

Functions and Transformations

Re( ) is the real part of a complex number, for example, $\mathrm{Re}(i^r_n + ji^j_n) = i^r_n$

Im( ) is the real part of a complex number, for example, $\mathrm{Im}(i^r_n + ji^j_n) = i^j_n$

| | is the magnitude of a complex number, for example, $|v_n| = [(v^r_n)^2+(v^j_n)^2]^{1/2}$

abs( ) is the absolute value function.

The transformation from rectangular to polar coordinates for voltage is:

$v^r_n = |v_n|\cos(\theta_n)$

$v^j_n = |v_n|\sin(\theta_n)$

$(v^r_n)^2 + (v^j_n)^2 = [|v_n|\sin(\theta_n)]^2 + [|v_n|\cos(\theta_n)]^2 = |v_n|^2[\sin(\theta_n)^2 + \cos(\theta_n)^2] = |v_n|^2$

We drop the bus index n and let θ be the voltage angle and δ be the current angle. For real power, $p = v^r i^r + v^j i^j = |v||i|\cos(\theta - \delta)$ and for reactive power, $q = v^j i^r - v^r i^j = |v||i|\sin(\theta - \delta)$ where $\theta - \delta$ is the power angle.

Parameters

r_{nmk} is the resistance of transmission element k.

x_{nmk} is the inductance of transmission element k.

s^{max}_k is the apparent power thermal limit on transmission element k at both terminal buses.

i^{max}_k is the current limit on transmission element k at both terminal buses.

$\theta^{\min}_{nm}$, θ^{max}_{nm} are the maximum and minimum voltage angle differences between n and m

$p^{\min}_n$, p^{max}_n are the maximum and minimum real power for generator n

$q^{\min}_n$, q^{max}_n are the maximum and minimum reactive power for generator n

$C_1 = (c^1_1, \ldots, c^1_N)^T$ and $C_2 = (c^2_1, \ldots, c^2_N)^T$ are vectors of linear and quadratic objective function cost coefficients respectively.

$$i_{nmk} = a_{nmk}^2 y_{nmk} v_n - a_{nmk} y_{nmk} v_m$$

$$i_{mnk} = -a_{nmk} y_{nmk} v_n + y_{nmk} v_m$$

For the phase shifting transformer (PAR) with a phase angle shift of φ,

$v_m/v_n = t_{nmk} = a_{nmk}e^{j\varphi}$

$i_{nm}/i_{mn} = t_{nmk}^* = -a_{nmk}e^{-j\varphi}$

The current-voltage equations for the phase shifting transformer k between buses n and m are:

$i_{nmk} = a_{nmk}^2 y_{nmk} v_n$ - $t_{nmk}^* y_{nmk} v_m$

$i_{mnk} = -t_{nmk} y_{nmk} v_n + y_{nmk} v_m$

Admittance Matrix. If there are no transformers or flexible AC transmission system (FACTS) devices, G is positive semidefinite and B is negative semidefinite. A matrix where $y_{nn} \geq \text{abs}(\sum_m y_{nm})$ is called diagonally dominant and strictly diagonally dominant if $y_{nn} > \text{abs}(\sum_m y_{nm})$.

If there are no transformers and $y_n = 0$, G and B are weighted Laplacian matrices of the undirected weighted graph that describes the transmission network. Much is known about the weighted Laplacian matrices. Y is a complex weighted Laplacian matrix. If $\text{abs}(y_n) << \text{abs}(\sum_m y_{nm})$ close to 0, we will call Y an approximate complex weighted Laplacian.

$Y = G+jB$. G and B are real symmetric diagonally dominant matrices. A symmetric diagonally dominant matrix has a symmetric factorization, for example, B = UU^T where each column of U has at most two non-zeros and the non-zeroes have the same absolute value.

If there are transformers and FACTS devices, let $\underline{y}_{nmk}$ be y_{nmk} , $a_{nmk}^2 y_{nmk}$, $t_{nmk}^* y_{nmk}$, or $-t_{nmk} y_{nmk} v_n$ as appropriate off-diagonal element, then $y_{nn} = y_n + \sum_{k,m} \underline{y}_{nmk}$, $y_{nm} = \sum_k \underline{y}_{nmk}$, and Y is the matrix $[y_{nm}]$. If there are only ideal in-phase transformers, the Y matrix is symmetric. If there are phase shifting transformers, the symmetry of the Y matrix is lost.

For large problems, the admittance matrix, $Y = G+jB$, is usually sparse. The density of both G and B is $(N+2K')/N^2$ where K' is the number of off-diagonal non-zero entries (the aggregate of multiple transmission elements between adjacent busses) and n is the number of buses. For example, in a topology with 1000 buses and 1500 transmission elements, G and B would have a density of $(1000+3000)/1000^2 = .004$. The lowest density for a connected network is the spanning tree. It has N-1 transmission elements and the density is $(N+2(N-1))/N^2$. For large sparse systems, $(N+2(N-1))/N^2 \approx 3/N$.

3. AC Flow Equations

Kirchhoff's current law. Kirchhoff's current law requires that the sum of the currents injected and withdrawn at bus n equal zero:

$$\mathrm{i}_n = \sum_k \mathrm{i}_{nmk} \qquad (1)$$

If we define the ground to be bus 0, current to ground to be $y_n(v_n - v_0)$ and $v_0 = 0$, we have:

$$\mathrm{i}_n = \sum_k y_{nmk}(v_n - v_m) + y_n v_n \qquad (2)$$

Current is a linear function of voltage. Rearranging,

$$\mathrm{i}_n = v_n(y_n + \sum_k y_{nmk}) - \sum_k y_{nmk} v_m \qquad (3)$$

Expanding, we obtain

$$\mathrm{i}_{nmk} = y_{nmk}(v_n - v_m) = g_{nmk}(v^r{}_n - v^r{}_m) - b_{nmk}(v^j{}_n - v^j{}_m) + \mathrm{j}(b_{nmk}(v^r{}_n - v^r{}_m) + g_{nmk}(v^j{}_n - v^j{}_m))$$

where $\mathrm{i}^r{}_{nmk} = g_{nmk}(v^r{}_n - v^r{}_m) - b_{nmk}(v^j{}_n - v^j{}_m)$ and

$\mathrm{i}^j{}_{nmk} = b_{nmk}(v^r{}_n - v^r{}_m) + g_{nmk}(v^j{}_n - v^j{}_m)$

In matrix notation, the IV flow equations in terms of current (I) and voltage (V) in (3) are

$$I = YV = (G + jB)(V^r + jV^j) = GV^r - BV^j + j(BV^r + GV^j) \qquad (4)$$

where $\mathrm{I}^r = GV^r - BV^j$ and $\mathrm{I}^j = BV^r + GV^j$

In another matrix format, (4) is

$I = (I^r, I^j) = \underline{Y}(V^r, V^j)^{\mathrm{T}}$ or

$$I = (I^r, I^j) = \begin{bmatrix} G & -B \\ B & G \end{bmatrix} \begin{bmatrix} V^r \\ V^j \end{bmatrix} \quad \text{where } \underline{Y} = \begin{bmatrix} G & -B \\ B & G \end{bmatrix}$$

If a and φ are constant, the $I = YV$ equations are linear. If not, the linearity is lost since the some elements of the Y matrix are variable. Discrete setting on the transmission assets like PARs and FACTS devices can be modeled with binary variables, but otherwise retain linearity.

Power flow equations. The traditional rectangular real power (P), reactive power (Q) and voltage (V) power flow equations (PQV) are

$$S = P + jQ = \mathrm{diag}(V)I^* = \mathrm{diag}(V)[YV]^* = \mathrm{diag}(V)Y^*V^* \qquad (5)$$

The power injections are

$$S = V \bullet I^* = (V^r + jV^j) \bullet (\mathrm{I}^r - j\mathrm{I}^j) = (V^r \bullet I^r + V^j \bullet I^j) + j(V^j \bullet I^r - V^r \bullet I^j) \qquad (6)$$

where

$$P = V^r \bullet I^r + V^j \bullet I^j \qquad (7)$$

$$Q = V^j \bullet I^r - V^r \bullet I^j \qquad (8)$$

The PQV power flow equations, (5) and (6), are quadratic. The IV flow equations (4) are linear.

4. Generator, Load, Transmission, Voltage, and Angle Constraints

We present the bus level constraints in terms of the current and voltage at each bus.

Generator and Load Constraints. The standard but simplified representation of generator is used in most ACOPF formulation. The lower and upper bound constraints for generation (injection) and load (withdrawal) at bus n are:

$$p^{\min}{}_n \leq p \leq p^{max}{}_n \tag{9}$$

$$q^{\min}{}_n \leq q \leq q^{max}{}_n \tag{10}$$

In terms of v and i,

$$v^r{}_n i^r{}_n + v^j{}_n i^j{}_n \leq p^{max}{}_n \tag{11}$$

$$p^{min}{}_n \leq v^r{}_n i^r{}_n + v^j{}_n i^j{}_n \tag{12}$$

$$v^j{}_n i^r{}_n - v^r{}_n i^j{}_n \leq q^{max}{}_n \tag{13}$$

$$q^{\min}{}_n \leq v^j{}_n i^r{}_n - v^r{}_n i^j{}_n \tag{14}$$

Inequalities (11)-(14) along with other thermal constraints on equipment enforced at each generator bus constitute a four-dimensional 'D-curve' in the IV space; (11)-(14) are non-convex constraints. Additional D-curves defining the tradeoff between real and reactive power are circles in the p-q plane, are their intersection of constitutes a convex set and can be easily linearized. Since here we model a single period, ramp rates are unnecessary. The model is easily extended to multiple periods.

Voltage constraints. The two constraints that limit the voltage magnitude in rectangular coordinates at each bus n are

$$(v^r{}_n)^2 + (v^j{}_n)^2 \leq (v^{max}{}_n)^2 \tag{15}$$

$$(v^{\min}{}_n)^2 \leq (v^r{}_n)^2 + (v^j{}_n)^2 \tag{16}$$

Again, each nonlinear inequality involves only the voltages at bus n. The inequality depicted in (15) is a convex constraint but the one in (16) is not.

Line Flow Constraints. The thermal transmission limit on k, $s^{max}{}_k$, is a based on the asset materials ability to withstand temperature increases. As current increases, lines sag and equipment may be damaged by overheating. Transmission assets generally have three progressively larger thermal ratings: steady-state, 4 hour and 30 minute. The apparent power at n on k to m is:

$$\mathrm{s}_{nmk} = v_n \mathrm{i}_{nmk} = v_n y_{nmk}(v_n - v_m) = v_n y_{nmk} v_n - v_n y_{nmk} v_m.$$

The thermal limit on s_{nmk} is

$$(s^r{}_{nmk})^2 + (s^j{}_{nmk})^2 = |s_{nmk}|^2 \leq (s^{max}{}_k)^2 \tag{17}$$

These constraints are quadratic in $\mathrm{s}^r{}_{nmk}$ and $\mathrm{s}^j{}_{nmk}$ and quartic in $v^r{}_n$, $v^j{}_n$, $v^r{}_m$, $v^j{}_m$.

For the IV formulation, the constraints that limit the current magnitude in rectangular coordinates at each bus n on k are

$$(i^r{}_{nmk})^2 + (i^j{}_{nmk})^2 \leq (i^{max}{}_{nmk})^2 \tag{18}$$

Again, the nonlinearities are convex quadratic and isolated to the complex current at the bus. The inequality in (18) is a convex constraint. Generally, the maximum currents, i^{max}_{nmk}, are determined by material science studies. Limiting current may be a better physical constraint then limiting apparent power.

Voltage Angle Constraints. The power flowing over an AC line is approximately proportional to the sine of the voltage phase-angle difference at the receiving and transmitting ends. For stability reasons, the terminal-buses voltage-angle differences on transmission elements can be constrained as follows:

$$\theta^{\min}_{nm} \leq \theta_n - \theta_m \leq \theta^{max}_{nm}. \qquad (19)$$

In the rectangular formulation, the arctan function appears in some constraints.

$$\theta^{min}_{nm} \leq \arctan(v^j_n / v^r_n) - \arctan(v^j_m / v^r_m) \leq \theta^{max}_{nm}. \qquad (20)$$

5. ACOPF Formulation

Objective function. For a generator, the cost of generation is a function of the apparent power generated, $c(S) = c_P(P) + c_Q(Q)$, where $S = (P^2+Q^2)^{1/2}$. Most of the literature assumes $c_Q(Q) = 0$ and $c_P(P)$ is quadratic in P. There is little empirical evidence for these assumptions and they may have been made to fit the nonlinear solver. There is evidence that the cost functions are better approximated by piecewise linear functions. With binary variables and linear functions, there is no need to assume that the generator cost function is monotonic non-decreasing. All ISOs use piecewise linear functions. If we assume that the cost of reactive power is small compared to the cost of real power and if the cost function, c(S), is linear in S, an approximation of c(S) is

$$c(S) \approx c_P(P) + c_Q(|Q|).$$

For most generators, the normal operating range is $|Q| < 0.3P$. The absolute value function, $|Q|$, can be made a linear function with the transformation $|Q| = Q^+ + Q^-$ and $Q = Q^+ - Q^-$ where $Q^+, Q^- \geq 0$. By virtue of the minimization and the constraint formulation, $Q^+ \bullet Q^- = 0$.

If there is a value function for load or demand, d(S), the objective function is to maximize the market benefits from trade, d(S) – c(S). Similar arguments hold to simplify d(S) but are beyond the scope of this paper.

Rectangular Power Voltage Formulation The rectangular power voltage (rectangular PQV) ACOPF (rectangular ACOPF-PQV) formulation is:

Network-wide objective function: Min $c(S)$ (21)

Network-wide constraint: $P + jQ = S = V \bullet I^* = V \bullet Y^* V^*$ (22)

Bus-specific constraints

$$P^{min} \leq P \leq P^{max} \qquad (23)$$

$$Q^{min} \leq Q \leq Q^{max} \qquad (24)$$

$$V^r \bullet V^r + V^j \bullet V^j \leq (V^{max})^2 \qquad (25)$$

$$(V^{min})^2 \leq V^r \bullet V^r + V^j \bullet V^j \qquad (26)$$

$$(s_{nmk})^2 \leq (s^{max}_k)^2 \quad \text{for all k} \qquad (27)$$

$$\theta^{min}_{nm} \leq \arctan(v^j_n / v^r_n) - \arctan(v^j_m / v^r_m) \leq \theta^{max}_{nm} \quad (28)$$

In this formulation, network-wide constraints in (22) are 2N quadratic equalities that apply throughout the network; the bus-specific constraints in (23)-(24) are simple variable bounds at each bus; the constraints in (25) are convex quadratic inequalities at each bus; the constraints in (26) are nonconvex quadratic inequalities at each bus; the constraints in (27) are quartic inequalities in v an i at each bus; and the constraints in (28) are nonconvex inequalities.

Polar Power Voltage Formulation The polar power voltage (polar *PQV*) ACOPF (polar ACOPF-*PQV*) replaces quadratic equality constraints in (22) with the polar formulation of (22):

Network-wide constraints:

$$p_n = \sum_{mk} v_n v_m (g_{nmk} \cos\theta_{nm} + b_{nmk} \sin\theta_{nm}) \quad \text{for all } n \quad (29)$$

$$q_n = \sum_{mk} v_n v_m (g_{nmk} \sin\theta_{nm} - b_{nmk} \cos\theta_{nm}) \quad \text{for all } n \quad (30)$$

(25) - (26) is replaced by $V^{min} \leq V \leq V^{max}$ (31) -(32)

(28) is replaced by $\theta^{min}_{nm} \leq \theta_n - \theta_m \leq \theta^{max}_{nm}$. for all n, m (33)

In this formulation, the network-wide (29)-(30) are 2N nonlinear equality constraints with quadratic terms and sine and cosine functions that apply throughout the network. The arctan functions disappear in the angle difference constraints.

Rectangular ACOPF-IV formulation. The rectangular ACOPF-IV formulation is:

Network-wide objective function: Min c(S) (34)

Network-wide constraint: $I = YV$ (35)

Bus-specific constraints:

$$P = V^r \bullet I^r + V^j \bullet I^j \leq P^{max} \quad (36)$$

$$P^{min} \leq P = V^r \bullet I^r + V^j \bullet I^j \quad (37)$$

$$Q = V^j \bullet I^r - V^r \bullet I^j \leq Q^{max} \quad (38)$$

$$Q^{min} \leq Q = V^j \bullet I^r - V^r \bullet I^j \quad (39)$$

$$V^r \bullet V^r + V^j \bullet V^j \leq (V^{max})^2 \quad (40)$$

$$(V^{min})^2 \leq V^r \bullet V^r + V^j \bullet V^j \quad (41)$$

$$(i^r_{nmk})^2 + (i^j_{nmk})^2 \leq (i^{max}_k)^2 \quad \text{for all k} \quad (42)$$

$$\theta^{min}_{nm} \leq \arctan(v^j_n / v^r_n) - \arctan(v^j_m / v^r_m) \leq \theta^{max}_{nm} \quad (43)$$

In this formulation, the constraints in (35) are 2N linear equality constraints that apply throughout the network. This is in contrast to the *PQV* formulations where quadratic and trigonometric constraints apply throughout the network and linear constraints are isolated at each bus. The constraints in (36) – (39) are locally quadratic and non-convex. The constraints in (40) and (42) are convex locally quadratic inequality constraints, but the ones in (41) are non-convex locally quadratic inequality constraints. Overall, the constraint set is still non-convex, but it would appear that this formulation may be easier to solve since the nonlinearities are isolated to each bus and each transmission element, while the constraint that

applies throughout the network is linear. The constraint in (43) could be eliminated by using the constraints in (42) as a surrogate and the problem becomes locally quadratic with linear network equations.
The *IV* formulation has 6N variables (P, Q, V^r, V^j, I^r, I^j) and the polar *PQV* has 4N variables (P, Q, $|V|$, Θ).

6. The Linear Approximations to the I*V* Formulation

A judicious choice of constraint formulations may produce better approximations. We take two approaches to constraint formulation. If the constraint is convex, we add linear cutting planes to remove from the linear feasible region points that are infeasible in the nonlinear formulation. If the constraint is non-convex, we use the first order Taylor series approximation to the constraint and update it after optimal iteration of the linear program.

Taylor's series. The Taylor series approximation for a function of n variables, $f(X)$ where $X = (x_1, \ldots, x_N)$ is

In matrix notation,

$$f(X) = f(\underline{X}) + f'(\underline{X})(X-\underline{X}) + (X-\underline{X})^T f''(\underline{X})(X-\underline{X}) + o[(X-\underline{X})\bullet(X-\underline{X})\bullet(X-\underline{X})]$$

where $f'(\underline{X}) = [(\partial f(\underline{X})/\partial x_1), \ldots \partial f(\underline{X})/\partial x_N)]^T$ is the Jacobian vector and

$f''(\underline{X}) = [\partial^2 f(\underline{X})/\partial x_n \partial x_m]$ is the Hessian matrix.

Convexity. If $f(X)$ is convex and differentiable, $f(X) \geq f(\underline{X}) + f'(\underline{X})^T(X-\underline{X})$ or

$$f(X) - f'(\underline{X})^T X \geq f(\underline{X}) - f'(\underline{X})^T \underline{X}$$

and $f''(\underline{X})$ is positive semidefinite.

If $f(X) \leq f^{up}$ and $f(X) \geq f(\underline{X}) - f'(\underline{X})^T \underline{X} + f'(\underline{X})^T X$,

$$f(\underline{X}) - f'(\underline{X})^T \underline{X} + f'(\underline{X})^T X \leq f^{up}$$

is a relaxation of the convex constraint that includes the feasible set, but excludes $\underline{X}$.

If $f^{low} \leq f(X)$, and $f(X) \geq f(\underline{X}) - f'(\underline{X})^T \underline{X} + f'(\underline{X})^T X$,

$$f^{low} \leq f(\underline{X}) - f'(\underline{X})^T \underline{X} + f'(\underline{X})^T X$$

is not a conservative linear approximation and not a convex constraint and we may need to limit the linear approximation to small excursions from $f(\underline{X})$, for example,

$f^{low} = \max\{f^{low}, rf(\underline{X})\}$ where r is the acceptable range of the approximation.

Linear Voltage Approximations. We can linearize the voltage constraint with a first order Taylor's series approximation about ($\underline{v}^r$, $\underline{v}^j$). at each bus n, the voltage magnitude approximation is

$$v^r_n v^r_n + v^j_n v^j_n \approx \underline{v}^r_n \underline{v}^r_n + \underline{v}^j_n \underline{v}^j_n + 2\underline{v}^r_n(v^r_n - \underline{v}^r_n) + 2\underline{v}^j_n(v^j_n - \underline{v}^j_n) \quad (44)$$

Collecting terms, we obtain

$$v^r_n v^r_n + v^j_n v^j_n \approx 2\underline{v}^r_n v^r_n + 2\underline{v}^j_n v^j_n - \underline{v}^r_n \underline{v}^r_n - \underline{v}^j_n \underline{v}^j_n \quad (45)$$

The Hessian is

$$\partial^2(|v_n|^2)/\partial v^r_n \partial v^j_n = \partial^2(v^r_n v^r_n + v^j_n v^j_n)/\partial v^r_n \partial v^j_n = \begin{array}{cc} v^r_n & v^j_n \\ \left[\begin{array}{c} 2 \\ 0 \end{array}\right. & \left.\begin{array}{c} 0 \\ 2 \end{array}\right] \end{array} \begin{array}{c} v^r_n \\ v^j_n \end{array} \quad (46)$$

Since the Hessian is positive definite, the voltage magnitude function is convex, (40) is a convex constraint and (41) is a non-convex constraint.

Iterative Voltage Constraints. Let $\underline{v}^r_n$, $\underline{v}^j_n$ be the optimal solution to the LP approximation that may violate the nonlinear constraint (40). We can add a linear constraint to create an outer bound on the constraint set.

Let a = $\underline{v}^j_n / \underline{v}^r_n$,

$\underline{v}^r_n \underline{v}^r_n + \underline{v}^j_n \underline{v}^j_n = \underline{v}^r_n \underline{v}^r_n + a^2 \underline{v}^r_n \underline{v}^r_n = (1+a^2)\underline{v}^r_n \underline{v}^r_n$.

If $\underline{v}^r_n \underline{v}^r_n + \underline{v}^j_n \underline{v}^j_n = (1+ a^2)\underline{v}^r_n \underline{v}^r_n > (v^{max}_n)^2$,

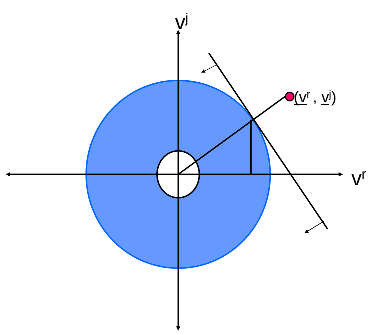
reset $\underline{v}^r_n = \mathrm{sign}(a)[(v^{max}_n)^2/(1+a^2)]^{1/2}$ and $\underline{v}^j_n = a\underline{v}^r_n$

Now $\underline{v}^j_n$, $\underline{v}^r_n$ is a point on the maximum voltage constraint (40).

add the linear constraint: $\underline{v}^r_n v^r_n + \underline{v}^j_n v^j_n \leq (v^{max}_n)^2$ (47)

(47) cuts off the linear program solution, and is tangent to and contains (40). Figure 1 illustrates the new constraint where the shaded area is the non-convex feasible region of the maximum and minimum voltage at a bus. Since the maximum voltage constraint is convex, the outer approximation linear constraints on maximum voltage contain the feasible region.

Figure 1. Adding a maximum-voltage linear constraint.



The outer approximation can accumulate iteratively and/or can be generated as a part of preprocessing without eliminating any part of the feasible solution. We can create a linear approximation by the following process. We can choose several points on the boundary of the maximum voltage constraint and add the outer linearization to the formulation in the preprocessing.

Preprocessed Voltage Constraints. We can start with simple linear bounds on the maximum voltage, if $v^j_n = 0$, the outer linearization constraint at the voltage angles 0 and π,

$-v^{max}_n \leq v^r_n \leq v^{max}_n$ (48)

if $v^r_n = 0$, the outer linearizations at the voltage angles π/2 and 3π/2 are:

$-v^{max}{}_n \leq v^j{}_n \leq v^{max}{}_n$ (49)

We also bound at π/4 and 5π/4, $v^r{}_n = v^j{}_n$, at the boundary:

$$v^r{}_n v^r{}_n + v^j{}_n v^j{}_n = 2\, v^r{}_n v^r{}_n = (v^{max}{}_n)^2$$

the tangent points are $v^r{}_n = \pm v^{max}{}_n/2^{1/2}$ and $v^j{}_n = \pm 2^{-1/2} v^{max}{}_n$

The constraints at π/4 and 5π/4 are:

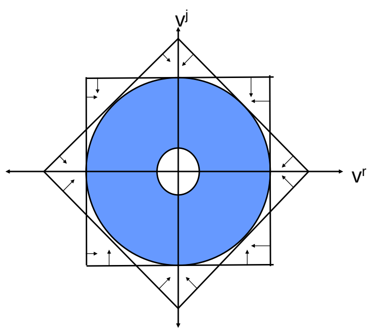
$-2^{1/2} v^{max}{}_n \leq v^r{}_n + v^j{}_n \leq 2^{1/2} v^{max}{}_n$ (50)

In a similar manner, at 3π/4 and 7π/4, with $v^r{}_n = -v^j{}_n$, the constraints are:

$-2^{1/2} v^{max}{}_n \leq v^r{}_n - v^j{}_n \leq 2^{1/2} v^{max}{}_n$ (51)

Inequalities (48), (49), (50) and (51) create polygon constraints as shown in Figure 2.

Figure 2. Regular 8-polygon bounds on voltage variables



Non-Convex Minimum Voltage Constraints. For a minimum voltage constraint, since it is non-convex, the linear approximation is more problematic. This may be a serious problem. Since higher losses occur at lower voltages, the natural tendency of the optimization will be toward higher voltages. It is an inner approximation and eliminates parts of the feasible region (see figure 4). Although this may be reasonable, we cannot accumulate these constraints and should relinearize after each linear program pass.

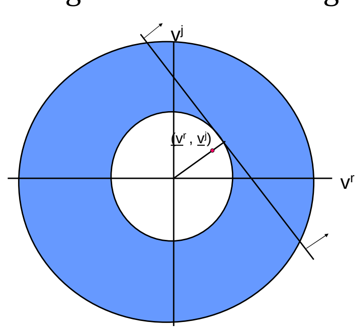
Let $\underline{v}^r{}_n$, $\underline{v}^j{}_n$ be the optimal solution to the LP approximation and assume it violates the nonlinear nonconvex constraint (41). Let $\underline{v}^j{}_n = a\underline{v}^r{}_n$. If $\underline{v}^r{}_n \underline{v}^r{}_n + \underline{v}^j{}_n \underline{v}^j{}_n < (v^{min}{}_n)^2$,

reset $\underline{v}^r{}_n = \mathrm{sign}(a)[(v^{min}{}_n)^2/(1+a^2)]^{1/2}$ and $\underline{v}^j{}_n = a\underline{v}^r{}_n$

Add the linear constraint:

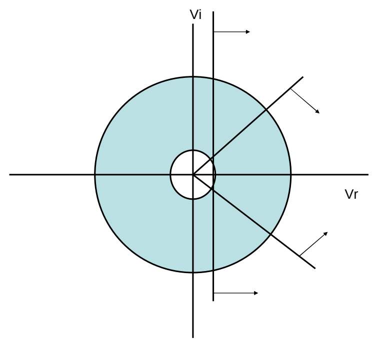
$\underline{v}^r{}_n v^r{}_n + \underline{v}^j{}_n v^j{}_n \geq (v^{min}{}_n)^2$ (52)

Figure 4. Adding a minimum-voltage linear constraint.



If we have voltage angle constraints, there may be a better idea. If there are voltage angle constraints, a linear representation of the voltage constraints may be at hand. If we add a constraint, $v^{\min}_n \leq v^r_n$. Let $v^j_n - a_1 v^r_n \leq 0$ and $v^j_n + a_2 v^r_n \leq 0$ represent voltage angle constraints. The resulting constraint set is convex, see Figure 5 and contains most of the nonconvex constraint set. The stability of the power system is a function of the angle differences.

Figure 5. Minimum-voltage with voltage angle constraints



When combined with a maximum voltage linearization, the approximation formulation could be preprocessed to obtain a good fit to the nonlinear constraints. Since the natural process of optimization pushes the voltage higher to avoid losses, the minimum voltage constraint is only approximated when it is violated.

Linear Approximation of Thermal Transmission Constraints

The constraints that limit MVA flow, (27), and the constraint on current flow perform similar functions. There are no minimum current constraints. To eliminate subscript clutter, we will drop the subscripts n and m, that is, i^r_{nmk}, i^j_{nmk} becomes i^r_k, i^j_k. Similar to the maximum voltage constraints, we can linearize the current constraint with a first order Taylor's series approximation about (i^r_{nmk}, i^j_{nmk})

$$i^r_k i^r_k + i^j_k i^j_k \approx i^r_k i^r_k + i^j_k i^j_k + 2 i^r_k (i^r_k - i^r_k) + 2 i^j_k (i^j_k - i^j_k) \qquad (53)$$

Collecting terms, we obtain the linear approximation is:

$$i^r_k i^r_k + i^j_k i^j_k \approx 2 i^r_k i^r_k + 2 i^j_k i^j_k - i^r_k i^r_k - i^j_k i^j_k \qquad (54)$$

The Hessian is

$$H(i_k) = \partial^2(|i_k|^2)/\partial i^r_k \partial i^j_k = \partial^2(i^r_k i^r_k + i^j_k i^j_k)/\partial i^r_k \partial i^j_k = \begin{array}{cc} i^r_k & i^j_k \\ \left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}\right] & \end{array} \begin{array}{c} i^r_k \\ i^j_k \end{array} \qquad (55)$$

Since the Hessian is positive definite, the current magnitude function is convex in (i^r_k, i^j_k). In the IV formulation, the current magnitude is functionally the same as the voltage magnitude.

Preprocessed Current Constraints. We can start with simple linear bounds on the maximum current, if $i^j_k = 0$, the outer linearization constraint the current angles 0 and π,

$$-i^{max}_k \leq i^r_k \leq i^{max}_k \qquad (56)$$

If $i^r{}_k = 0$, the outer linearizations at the current angles π/2 and 3π/2 are:

$$-i^{max}{}_k \leq i^j{}_k \leq i^{max}{}_k \qquad (57)$$

Inequalities (56) and (57) create box constraints as shown in Figure 2.
We also bound at π/4 and 5π/4, $i^r{}_k = i^j{}_k$, at the boundary:

$$i^r{}_k i^r{}_k + i^j{}_k i^j{}_k = i^r{}_k i^r{}_k = (i^{max}{}_k)^2/2.$$

The constraints at π/4 and 5π/4 are:

$$-2^{1/2} i^{max}{}_k \leq i^r{}_k + i^j{}_k \leq 2^{1/2} i^{max}{}_k \qquad (58)$$

At 3π/4 and 7π/4, with $i^r{}_k = -i^j{}_k$ the constraints are:

$$-2^{1/2} i^{max}{}_k \leq i^r{}_k - i^j{}_k \leq 2^{1/2} i^{max}{}_k \qquad (59)$$

Inequalities (56), (57), (58) and (59) create the polygon constraints shown in Figure 2.

Iterative Current Constraints. Let $\underline{i}^r{}_k$, $\underline{i}^j{}_k$ be the optimal solution to the LP approximation that may violate the nonlinear constraint. We develop the maximum current constraints in a similar manner to the maximum voltage constraints. We can add a linear constraint to create an outer bound on the constraint set.

Let $a = \underline{i}^j{}_k / \underline{i}^r{}_k$. If $\underline{i}^r{}_k \underline{i}^r{}_k + \underline{i}^j{}_k \underline{i}^j{}_k > (i^{max}{}_k)^2$,

reset $\underline{i}^r{}_k = \text{sign}(a)[(i^{max}{}_k)^2/(1+a^2)]^{1/2}$ and $\underline{i}^j{}_k = a\underline{i}^r{}_k$

add the linear constraint: $\underline{i}^r{}_k i^r{}_k + \underline{i}^j{}_k i^j{}_k \leq (i^{max}{}_k)^2$ (60)

If we change the voltage variables to current variables, Figure 1 illustrates the new constraint where the outer circle defines the boundary of the convex feasible region (for current there is no inner circle minimum constraint). Since the max current constraint is convex, the outer approximation linear constraints on maximum current contain the nonlinear feasible region.

The outer approximation can accumulate iteratively and/or can be generated as a part of prepossessing without eliminating any part of the feasible solution. We can create a linear approximation by the following process. We can choose several points on the boundary of the maximum current constraint and add the outer linearization to the formulation in the prepossessing. As experience with this model increases, the constraints can be chosen based on experience and specific system behavior.

Voltage-angle difference constraints serve the purpose of limiting the line flows, but in rectangular coordinates are not convex. Limiting current flow has a similar effect, but the current flow limits are convex. Therefore, substituting current for angle constraints may be a better computational formulation.

Real Power Constraints. Constraints, (36) - (37), are real power generator constraints. At each bus n, p_n is the sum of two hyperbolas in the real and imaginary planes. We can linearize them with a first order approximation as follows:

$$p_n = v^r{}_n i^r{}_n + v^j{}_n i^j{}_n$$

$$\approx \underline{v}^r \bullet \underline{i}^r + \underline{v}^j \bullet \underline{i}^j + \underline{v}^r \bullet i^r + \underline{v}^j \bullet i^j + v^r \bullet \underline{i}^r + v^j \bullet \underline{i}^j - (\underline{v}^r \bullet \underline{i}^r + \underline{v}^j \bullet \underline{i}^j + \underline{v}^r \bullet \underline{i}^r + \underline{v}^j \bullet \underline{i}^j) \qquad (61)$$

collecting terms, we obtain

$$p^{\approx} = \underline{v}^r{}_n i^r{}_n + \underline{v}^j{}_n i^j{}_n + v^r{}_n \underline{i}^r{}_n + v^j{}_n \underline{i}^j{}_n - (\underline{v}^r{}_n \underline{i}^r{}_n + \underline{v}^j{}_n \underline{i}^j{}_n) \quad (62)$$

The Hessian is

$$h(p_n)=\partial^2 p_n/\partial v_n \partial i_n = \begin{array}{cc} \begin{array}{cccc} v^r{}_n & v^j{}_n & i^r{}_n & i^j{}_n \end{array} & \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} v^r{}_n \\ v^j{}_n \\ i^r{}_n \\ i^j{}_n \end{array} \end{array} \quad (63)$$

The Hessian is an indefinite symmetric matrix with 2 eigenvalues equal to 1 and 2 eigenvalues equal to -1. Since the Hessian is indefinite, the real power function is non-convex. We add constraints:

$$P^{\min} - P^{r\min} \leq P^{\approx} \leq P^{max} + P^{rmax} \quad (64)$$

where $P^{r\min}$ and $P^{rmax} \geq 0$ with high objective function coefficients.

Reactive Power Constraints. Similarly, for reactive power,

$$q^{\approx} = \underline{v}^j{}_n i^r{}_n - \underline{v}^r{}_n i^j{}_n - v^r{}_n \underline{i}^j{}_n + v^j{}_n \underline{i}^r{}_n - (\underline{v}^j{}_n \underline{i}^r{}_n - \underline{v}^r{}_n \underline{i}^j{}_n) \quad (65)$$

The Hessian is

$$h(q) = \partial^2 q_n/\partial v_n \partial i_n = \begin{array}{cc} \begin{array}{cccc} v^r{}_n & v^j{}_n & i^r{}_n & i^j{}_n \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} v^r{}_n \\ v^j{}_n \\ i^r{}_n \\ i^j{}_n \end{array} \end{array} \quad (66)$$

The Hessian is an indefinite symmetric matrix consisting with 2 eigenvalues equal to 1 and 2 eigenvalues equal to -1. Since the Hessian is indefinite, the reactive power function is non-convex. We added the constraints:

$$Q^{\min} - Q^{r\min} \leq Q^{\approx} \leq Q^{max} + Q^{rmax} \quad (67)$$

where $Q^{r\min} \geq 0$ and $Q^{rmax} \geq 0$ with high objective function coefficients

7. Conclusions

Unlike the DC model that holds V constant, ignores reactive power and assumes small angle differences, the *IV* formulation solves a linear system of equations without decomposition, unnecessary constraints or omissions. The nonconvex constraints need careful consideration. It appears that the *IV* formulation and its linear approximations have promise to meet practical computational requirements. For problems that are solved repetitively with minor variations, there is considerable potential for individual parameter tuning and preprocessed constraints. For example, constraints on angles and past iteratively added constraints may be able to help. In addition, steady-state quadrature constraints on generators are linear in the *IV* formulation and can be included in the formulation.

Additional formulation testing and computational testing are needed to determine commercial feasibility. If a linear IV approximation to the ACOPF proves promising, it can be embedded in the unit commitment models, optimal topology models and other formulations that use binary variables. This allows the use of MIP algorithms that are exceptionally fast and robust to better model the power markets.

References

Mary B. Cain, Richard P. O'Neill, Anya Castillo, "History of Optimal Power Flow and Formulations," FERC staff technical paper, December, 2012